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Three dimensional flow over a submerged object

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SUMMARY

An obstacle, represented by a delta function, is placed on the bed of a three dimensional stream and, as a result, a steady "V" shaped surface wave pattern develops downstream. The rate of decay of the transient motion is determined and existence and uniqueness of the steady pattern are established for all values of the stream velocity. Asymptotic expressions for the steady state wave amplitude, valid far from the obstacle, are obtained. Near the border of the wave pattern, expressions are obtained which are uniform in the polar angle.

Introduction

A three dimensional stream of constant depth h flows with constant velocity U. At time $t = 0$ a fixed obstacle, represented by a delta function, is introduced onto the stream bed. As a result, a transient wave motion is created on the surface, downstream from the obstacle, which eventually develops into a "V"-shaped steady pattern. The purpose of this article is

1) to obtain the rate of decay of the transient motion as $t \to \infty$;
2) to establish the existence and the uniqueness of the steady pattern, in particular for the critical velocity $U = (gh)^{\frac{1}{2}}$;
3) to obtain asymptotic expressions for the steady state wave amplitude which are valid for large distances from the obstacle, uniformly in the polar angle, in particular near the border of the "V" for critical, $U = (gh)^{\frac{1}{2}}$, and supercritical, $U > (gh)^{\frac{1}{2}}$, velocity.

The two dimensional version of this problem has been studied by Palm [10]. He obtains a solution by Fourier analysis, obtains the rate of decay of the transient motion as $t \to \infty$, and shows the existence and uniqueness of a steady wave pattern for non-critical flow velocities. A remarkable feature of the steady pattern is that, far from the obstacle, it is substantially different for subcritical than for supercritical velocities. For $U < (gh)^{\frac{1}{2}}$ there exists a sinusoidal "resonance" wave extending downstream to infinity without decay in amplitude whereas, for $U > (gh)^{\frac{1}{2}}$, the steady pattern decays exponentially in distance from the obstacle. This steady pattern is also discussed by Lamb [3] who constructs it directly using a radiation condition.

It is relevant to include a discussion of previous results for the closely related problem: to determine the transient and steady state wave motion resulting from the continued application of a pressure point, beginning at $t = 0$, on the surface of the three dimensional stream. In fact, since these results can be easily established for the present problem, we shall, in our treatment of it, lay stress on those aspects which have not received adequate attention in the problem of the pressure point, i.e., points 1)–3) above.

The two dimensional pressure point problem has been investigated by Stoker [1] who finds that the steady pattern is similar to that of Palm described above. Stoker, moreover, also investigated the case of critical velocity and showed that, as $t \to \infty$, the amplitude of the transient motion grows large. From this he concludes that a steady state pattern does not exist for critical speed. This surprising result raises the question of whether a steady pattern exists for critical speed in the three dimensional problem.

Havelock [4] has considered the three dimensional, *steady* pressure point problem (or equivalently: a ship moving with constant velocity U on water of finite depth h). For non-critical velocities he constructed directly a steady state solution, using a radiation condition, and was able to obtain its asymptotic description by the method of stationary phase. He, also, found a remarkable difference between the sub- and super-critical cases. For $U < (gh)^{\frac{1}{2}}$ the pattern is similar to Kelvin's ship wave pattern [5]: it has two sets of waves, transverse and diverging, contained in a "V" shaped sector behind the point which widens out to the downstream half plane as $U \to (gh)^{\frac{1}{2}}$. For $U > (gh)^{\frac{1}{2}}$ only the diverging waves are present in the sector and it narrows down to the half line in the flow direction as $U \to \infty$.

Cherkesov [11] has studied the time-dependent problem and has obtained an asymptotic description, valid far from the pressure point, of the transient wave front for all values of U. For subcritical speeds, Smorodin [12] has considered a pressure point of variable strength moving with variable speed.

In the asymptotic analysis of the pattern of Havelock described above, one encounters a difficulty near the border of the "V" shaped sector since the method of stationary phase doesn't yield an expression which is uniform in the polar angle. A similar situation occurs in the ship wave pattern of Kelvin, where since the pattern is $O(r^{-\frac{1}{2}})$ within the sector and $O(r^{-\frac{1}{3}})$ on its border, it is desirable to obtain an expression which shows the transition between these two orders of magnitude as the border is approached for fixed r. Ursell [7] has obtained such an expression, in terms of the Airy function, by utilizing the uniform asymptotic method of Chester, *et al.* [6]. Such an investigation does not seem to have been carried out for Havelock's pattern. It is required, in particular, for the critical and super-critical cases since, for subcritical speed, the analysis of Ursell is applicable.

In Theorem 1, we give the decay rate of the transient motion and thereby show the existence of a steady state pattern for all $U > 0$ including the critical value. In Theorem 3, we show that this pattern, which is similar to Havelock's, is unique and located downstream from the obstacle. Finally, we obtain uniform asymptotic expressions near the border of the steady state pattern. For $U < (gh)^{\frac{1}{2}}$ the result is similar to Ursell's for the ship wave pattern. For $U = (gh)^{\frac{1}{2}}$ we use a method of Bleistein [8] to obtain a uniform expression in terms of Fresnel integrals. In this case the border of the wave system is the axis transverse to the flow direction and, taking the limit in our expression for fixed r, we find that the wave amplitude tends to zero as the axis is approached. For $U > (gh)^{\frac{1}{2}}$ we use Chester's method to obtain a uniform expression in terms of the derivative of the Airy function and find, again by taking the limit for fixed r, that the limiting wave amplitude on the border of the system is $O(r^{-\frac{2}{3}})$.

The approach of arriving at the steady state motion via an initial value formulation is due to Stoker [1]. Its advantage is that the questions of existence and uniqueness of the steady state motion are settled in a natural way without having to impose a radiation condition.

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1. The initial boundary value problem

We consider an initially uniform potential stream flow $\Phi = Ux$ defined in the slab $-h \leqq y \leqq 0$, $-\infty < x, z < \infty$. At $t = 0$ an obstacle is introduced onto the stream bed $y = -h$ whose height off the bed is given by $\gamma(x, z)a(t)$ where $\gamma(x, z)$ is assumed continuous and of compact support and $a(t)$ which serves to create the obstacle has a continuous second derivative and satisfies $a(0) = a'(0) = a''(0) = 0$ and $a(t) = 1$, $t \geqq \varepsilon > 0$. For $t \geqq 0$ we seek a solution in the form

$$\Phi(x, y, z, t) = Ux + \varphi(x, y, z, t)$$

defined in the region $-h + \gamma(x, z)a(t) \leqq y \leqq \eta(x, z, t)$ where the free surface $y = \eta(x, z, t)$ is unknown a priori. Following Stoker [1], the boundary conditions are linearized about the planes $y = -h$ and $y = 0$ and one obtains the following initial boundary value problem for φ defined in the original slab:

$$\text{D.E.} \quad \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, \qquad t \geqq 0,\ -h < y < 0, \tag{1.1}$$

$$\text{I.C.} \quad \varphi(x, y, z, 0) = 0, \qquad -h \leqq y \leqq 0, \tag{1.2}$$

$$\text{I.C.} \quad \varphi_t(x, y, z, 0) = 0, \qquad -h \leqq y \leqq 0, \tag{1.3}$$

$$\text{B.C.} \quad g\eta + \varphi_t + U\varphi_x = 0, \qquad y = 0,\ t \geqq 0, \tag{1.4}$$

$$\text{B.C.} \quad \eta_t + U\eta_x - \varphi_y = 0, \qquad y = 0,\ t \geqq 0, \tag{1.5}$$

$$\text{B.C.} \quad U\gamma_x a + \gamma a' - \varphi_y = 0, \qquad y = -h,\ t \geqq 0. \tag{1.6}$$

By differentiation and elimination, conditions (1.4) and (1.5) are replaced by the single condition

$$\text{B.C.} \quad \varphi_{tt} + U^2\varphi_{xx} + 2U\varphi_{xt} + g\varphi_y = 0, \qquad y = 0,\ t \geqq 0. \tag{1.7}$$

The solution of the initial boundary problem (1.1), (1.2), (1.3), (1.6), (1.7) is obtained by using the Fourier transformation in the variables x and z.

$$\bar{\varphi}(\xi, y, \zeta, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x, y, z, t)\, \mathrm{e}^{-i\xi x - i\zeta z}\, dx\, dz.$$

Our problem then goes over into the following.

$$-\xi^2\bar{\varphi} + \bar{\varphi}_{yy} - \zeta^2\bar{\varphi} = 0, \qquad -h < y < 0,\ t > 0, \tag{1.1'}$$

$$\bar{\varphi}(\xi, y, \zeta, 0) = 0, \qquad t = 0, \tag{1.2'}$$

$$\bar{\varphi}_t(\xi, y, \zeta, 0) = 0, \qquad t = 0, \tag{1.3'}$$

$$\mathrm{i}\xi U\bar{\gamma}a + \bar{\gamma}a' - \bar{\varphi}_y = 0, \qquad y = -h,\ t \geqq 0, \tag{1.6'}$$

$$\bar{\varphi}_{tt} - \xi^2U^2\bar{\varphi} + 2\mathrm{i}\xi U\bar{\varphi}_t + g\bar{\varphi}_y = 0, \qquad y = 0,\ t \geqq 0. \tag{1.7'}$$

The general solution to (1.1′) is, setting $\rho = (\xi^2 + \zeta^2)^{\frac{1}{2}}$,

$$\bar{\varphi}(\xi, y, \zeta, t) = A(\xi, \zeta, t)\,\mathrm{e}^{\rho y} + B(\xi, \zeta, t)\,\mathrm{e}^{-\rho y}, \qquad t > 0. \tag{1.8}$$

Inserting (1.8) into (1.7′) and (1.6′) respectively, the following two equations result:

$$A_{tt} + B_{tt} - \xi^2 U^2 (A + B) + 2\mathrm{i}\xi U(A_t + B_t) + \rho g(A - B) = 0, \qquad t > 0, \tag{1.9}$$

$$\mathrm{i}\xi U \bar{\gamma} a(t) - \rho(A\,\mathrm{e}^{-\rho h} - B\,\mathrm{e}^{\rho h}) + \bar{\gamma} a = 0, \qquad t > 0. \tag{1.10}$$

Letting

$$C(\xi, \zeta, t) = A(\xi, \zeta, t) + B(\xi, \zeta, t), \tag{1.11a}$$

$$D(\xi, \zeta, t) = A(\xi, \zeta, t) - B(\xi, \zeta, t). \tag{1.11b}$$

we substitute C and D in place of A and B in (1.9) and (1.10) and then eliminate D, obtaining an equation for C.

$$C_{tt} + 2\mathrm{i}\xi U C_t + (\rho g \tanh \rho h - \xi^2 U^2)C = -\frac{g\bar{\gamma}}{\cosh \rho h}(\mathrm{i}\xi U a + a'). \tag{1.12}$$

From (1.8), (1.2′), and (1.3′) we deduce the following initial conditions for C:

$$C(\xi, \zeta, 0) = 0, \qquad C_t(\xi, \zeta, 0) = 0. \tag{1.13a, b}$$

The set of equations (1.12), (1.13a, b) constitute an initial value problem for the function $C(\xi, \zeta, t)$ in the variable t. Solving it by variation of parameters one gets

$$C(\xi, \zeta, t) = -\frac{g\bar{\gamma}}{\cosh \rho h}\int_0^t K(t-s)[\mathrm{i}\xi U a(s) + a'(s)]\,ds$$

where

$$K(t) = \mathrm{e}^{-\mathrm{i}\xi U t}(g\rho \tanh \rho h)^{-\frac{1}{2}} \sin[(g\rho \tanh \rho h)^{\frac{1}{2}} t].$$

Integrating by parts one obtains

$$C(\xi, \zeta, t) = -\frac{g\bar{\gamma}}{\cosh \rho h}\int_0^t a(s)[\mathrm{i}\xi U K(t-s) - K'(t-s)]\,ds.$$

Now setting $y = 0$ in (1.8), noting (1.11a), taking the inverse transform, and letting $\varepsilon \to 0$ in the definition of $a(t)$ one obtains

$$\varphi(x, 0, z, t) = -\frac{g}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{\gamma}\,\mathrm{e}^{i\xi x + i\zeta z}}{\cosh \rho h}\int_0^t [\mathrm{i}\xi U K(t-s) + K'(t-s)]\,ds\,d\xi\,d\zeta.$$

Then from (1.4), using the fact that $K(t)$ satisfies the homogeneous form of (1.12), we obtain the surface shape

$$\eta(x, z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{\gamma}\,\mathrm{e}^{i\xi x + i\zeta z}}{\cosh \rho h}\left[1 - \rho g \tanh \rho h \int_0^t K(t-s)\,ds\right] d\xi\,d\zeta. \tag{1.14}$$

Carrying out the s-integral, going over to the polar coordinates

$$h\xi = u\cos\varphi, \qquad x = hr\cos\theta,$$

$$h\zeta = u\sin\varphi, \qquad z = hr\sin\theta,$$

and setting $k^2 = U^2/gh$ one finally gets

$$\eta(x, z, t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} \frac{\bar{\gamma}u}{\cosh u}\left[\frac{k^2\cos^2\varphi}{k^2\cos^2\varphi - \tanh u/u}\right.$$
$$+ \frac{(\tanh u/u)^{\frac{1}{2}}\, e^{-it(g/h)^{\frac{1}{2}}(ku\cos\varphi + (u\tanh u)^{\frac{1}{2}})}}{2(k\cos\varphi + (\tanh u/u)^{\frac{1}{2}})}$$
$$\left. - \frac{(\tanh u/u)^{\frac{1}{2}}\, e^{-it(g/h)^{\frac{1}{2}}(ku\cos\varphi - (u\tanh u)^{\frac{1}{2}})}}{2(k\cos\varphi - (\tanh u/u)^{\frac{1}{2}})}\right] e^{iru\cos(\varphi-\theta)}\, du\, d\varphi. \tag{1.15}$$

2. The unsteady development – Existence of a steady state flow

We first establish the existence of a steady state flow as $t(gh)^{\frac{1}{2}} \to \infty$ for all values of $k > 0$ including the "critical" case $k = 1$ and, moreover, obtain an estimate on the rate of decay of the transient flow with time. We then obtain a representation of the steady state flow as a single integral.

We consider (1.15) for fixed values of φ and take the u-integral to be a complex integral along the positive real axis from zero to infinity. Since the entire integrand is analytic in u in a neighborhood of the real axis, as can be seen from (1.14), the path of integration, except for the origin, can be deformed within this neighborhood according to Cauchy's theorem. This allows one to deform the path of integration about singular points of each separate term of the u-integral as long as such a deformation is carried out in the same manner for all of the terms.

The only singularity of the time independent term is the pole $u(\varphi)$ defined implicitly by the equation

$$k^2\cos^2\varphi - \tanh u(\varphi)/u(\varphi) = 0, \qquad u(\varphi) \geqq 0. \tag{2.1}$$

Factoring, one sees that $u(\varphi)$ is also a pole of the first or second time dependent terms according as $\cos\varphi$ is greater or less than zero. The function $u(\varphi)$ plays an important role and we summarize its properties for future reference. Since $u(\varphi)$ has the double symmetry $u(\varphi) = u(\pi - \varphi)$ and $u(\varphi) = u(-\varphi)$ it is sufficient to consider it on the interval $[0, \pi/2]$. Its derivative is given by

$$u'(\varphi) = \frac{k^2 u(\varphi)\sin 2\varphi}{\tanh u(\varphi)/u(\varphi) - \operatorname{sech}^2 u(\varphi)} = \frac{k^2\sin 2\varphi}{|(\tanh u/u)'|_{u=u(\varphi)}}. \tag{2.2}$$

The behavior of $u(\varphi)$ differs according to the cases $k^2 < 1$, $k^2 = 1$, $k^2 > 1$. For $k^2 < 1$, $u(\varphi)$ is defined on the entire interval $[0, \pi/2]$ and $u(0) = u_0 > 0$ is defined by $k^2 = \tanh u_0/u_0$. From (2.2), $u'(0) = 0$ and $u(\varphi)$ is increasing on $[0, \pi/2]$. For $k = 1$ $u(\varphi)$ is again defined on $[0, \pi/2]$ however $u(0) = 0$. In the neighborhood of the origin one has from expanding (2.1) that $u'_+(0) = \sqrt{3}$ and $u(\varphi)$ is again increasing on $[0, \pi/2]$. For $k^2 > 1$, $u(\varphi)$ is defined on a proper sub-interval of $[0, \pi/2]$ namely $[\varphi_0, \pi/2]$ $\varphi_0 > 0$ where $\cos\varphi_0 = 1/k$ and moreover $u(\varphi_0) = 0$. From (2.2) one has $u'(\varphi) \to \infty$ as $\varphi \to \varphi_0$. In all cases as $\varphi \to \pi/2$ one must have $u \to \infty$ hence $\tanh u \simeq 1$ and thus $u(\varphi) \simeq (k\cos\varphi)^{-2}$. In particular $u(\varphi) \to \infty$ as $\varphi \to \pi/2$.

We now assert that one can deform the path of the u-integral (into a semi-circle of radius ε, say) about the pole $u = u(\varphi)$ in such a way that the time dependent terms in (1.15) will tend to zero for large time. In particular

Theorem 1. For $|\varphi| \leqq \pi/2$ $(\pi/2 < |\varphi| \leqq \pi)$ *if the path of u-integration is deformed below (above) the pole* $u = u(\varphi)$ *then both time dependent terms in* (1.15) *will be of order* $O[(t(g/h)^{\frac{1}{2}})^{-\frac{1}{2}}]$ *when* $k < 1$ *and* $O[(t(g/h)^{\frac{1}{2}})^{-\frac{1}{3}}]$ *when* $k \geqq 1$, *as* $t \to \infty$. *Hence for all* $k > 0$ *there exists a steady state flow which is given by*

$$\eta(r, \theta) = \frac{k^2}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{\infty} \frac{\bar{\gamma} u \cos^2 \varphi \, e^{iru\cos(\varphi - \theta)}}{\cosh u(k^2 \cos^2 \varphi - \tanh u/u)} du d\varphi \tag{2.3}$$

where the path of u-integration is deformed below (above) the pole $u = u(\varphi)$ *for* $|\varphi| < \pi/2$ $(\pi/2 < |\varphi| \leqq \pi)$.

Proof: see Appendix A.

We shall be interested in the evaluation of (2.3) for large r. In this, the details of the obstacle play a negligible role and we will henceforth take $\bar{\gamma} = 2\pi\overline{\delta(x, z)} = 1$. Moreover, we have the following result which enables us to significantly simplify (2.3).

Theorem 2. For large r, (2.3) *can be expressed as a single integral in the following form*

$$\eta(r, \theta) = -\operatorname{Im} \int_{-\pi/2}^{\pi/2} \frac{H(\varphi) \cot \varphi u(\varphi) u'(\varphi) \, e^{iru(\varphi)\cos(\varphi - \theta)}}{\cosh u(\varphi)} d\varphi + O(\log r/r), \tag{2.4}$$

where $H(\varphi) = \begin{cases} 1 & \text{if } \cos(\varphi - \theta) > 0 \text{ and } u(\varphi) \text{ is defined} \\ 0 & \text{otherwise.} \end{cases}$

Proof: see Appendix B.

This single integral representation for $\eta(r, \theta)$ is a convenient form for the application of the method of stationary phase and, as we will show, yields contributions which for each $k > 0$ are of a lower order than $O(\log r/r)$ and hence it will determine the dominant features of the wave pattern for large r. Accordingly, we will henceforth neglect the term of order $O(\log r/r)$.

It will be more convenient to go over to the new variable $u = u(\varphi)$ in (2.4). Using the properties of $u(\varphi)$ listed above, letting $u_0 = 0$ when $k \geqq 1$, and setting for $u \geqq u_0$ and $|\theta| \neq \pi/2$

$$p_{\pm}(u) = (u \tanh u)^{\frac{1}{2}} \pm \tan \theta (k^2 u^2 - u \tanh u)^{\frac{1}{2}}$$

where the branch cut is taken from u_0 to $-\infty$, (2.4) becomes

$$\eta(r, \theta) = \operatorname{Im} \int_{\infty}^{u_0} \frac{u G_{-}(u) \, e^{irk^{-1}p_{-}(u)\cos\theta}}{\cosh u[(k^2 u/\tanh u) - 1]^{\frac{1}{2}}} du - \operatorname{Im} \int_{u_0}^{\infty} \frac{u G_{+}(u) \, e^{irk^{-1}p_{+}(u)\cos\theta}}{\cosh u[(k^2 u/\tanh u) - 1]^{\frac{1}{2}}} du, \tag{2.5}$$

where $G_{\pm}(u) = \begin{cases} 1, & u^{-1}p_{\pm}(u)\cos\theta > 0 \\ 0, & \text{otherwise.} \end{cases}$

3. Uniqueness of the steady state flow

We now show that the location of the steady state wave pattern is uniquely determined and can exist only downstream from the obstacle. This result follows directly from (2.5) and hence no radiation condition needs to be imposed.

Since $\eta(r, -\theta) = \eta(r, \theta)$ as can be seen from (2.4), it is sufficient to take $\tan\theta \geqq 0$. Then $p_+(u)$ is strictly increasing on $[u_0, \infty)$ and by the Riemann–Lebesgue Lemma the second integral in (2.5) will be of order $O(r^{-1})$. As for $p_-(u)$, fixing θ one gets the following equation for its stationary points:

$$\tan\theta = \frac{[(k^2u/\tanh u) - 1]^{\frac{1}{2}}[\operatorname{sech}^2 u + (\tanh u/u)]}{2[k^2 - (\tanh u/u)] + u|(\tanh u/u)'|} \equiv g(u) \tag{3.1}$$

where $\begin{cases} u \geqq u_0, \; k \neq 1 \\ u > u_0, \; k = 1 \end{cases}$ and $|\theta| \neq \dfrac{\pi}{2}$.

Lemma 1. For any fixed $k > 0$, *if* $u = u_r$ *is a root of* (3.1) *with* $u_r > 0$, *then* $p_-(u_r) > 0$.

Proof: Estimating from (3.1) with $u = u_r$,

$$\begin{aligned}\tan\theta &< \frac{[(k^2u/\tanh u) - 1]^{\frac{1}{2}}[\operatorname{sech}^2 u + (\tanh u/u)]}{2[k^2 - (\tanh u/u)]} \\ &= \frac{(u\tanh u)^{\frac{1}{2}}}{[k^2u^2 - (u\tanh u)]^{\frac{1}{2}}}\,\frac{1}{2}\left[\frac{u}{\cosh u \sinh u} + 1\right] \\ &< \frac{(u\tanh u)^{\frac{1}{2}}}{[k^2u^2 - (u\tanh u)]^{\frac{1}{2}}}\end{aligned}$$

and the result follows.

We now prove the following uniqueness result:

Theorem 3. Any stationary point $u = u_r$ *of* $p_-(u)$, *with* $u_r > 0$, *is in the range of integration of the first integral of* (2.5) *for at most one value of* θ, *and that value lies in* $[0, \pi/2)$.

Proof: Since $0 \leqq \tan\theta < \infty$, each stationary point of $p_-(u)$, $u = u_r > 0$, i.e., root of (3.1), is obtained for two different values of θ: θ_1 in $[0, \pi/2)$ (downstream), and $\theta_2 = \theta_1 - \pi$ (upstream). However, in view of Lemma 1, one has $G_-(u_r) > 0$ for $\theta = \theta_1$ and $G_-(u_r) < 0$ for $\theta = \theta_2$. Hence u_r is in the range of integration only for θ_1.

In Theorem 3, the special cases $|\theta| = \pi/2$ and $u_r = 0$ are omitted, but we shall subsequently show that, for them, there is no contribution to the integral.

4. Flow at sub-critical speed

We now turn to the asymptotic evaluation of (2.5) for large r, taking first the subcritical case $0 < k < 1$. We find that the ordinary method of stationary phase yields a Kelvin-type wave system of two components existing in a sector $0 \leqq |\theta| < \theta_m^k$ downstream from the obstacle. Within the sector, that is, for $0 \leqq |\theta| \leqq \theta_m^k - \delta$ the wave amplitude is of order $O(r^{-\frac{1}{2}})$ uniformly in θ and on the border of the sector, $|\theta| = \theta_m^k$, is of order $O(r^{-\frac{1}{3}})$. In the transition region near the border, $\theta_m^k - \delta \leqq |\theta| \leqq \theta_m^k$, the ordinary method of stationary phase does not yield an expansion uniform in θ, so we use a method of Chester, *et al.* [6], obtaining a uniform expansion in terms of an Airy function. We show finally that for large depth, the pattern resembles the well-known "ship wave" pattern produced by a pressure point moving on water of infinite depth.

We first investigate the distribution of stationary points of $p_-(u)$ as θ varies. We recall that these are given by the roots of (3.1). Now in the present case $g(u)$ is positive $u > u_0$ and $g(u_0) = 0$. Writing $g(u)$ in the form

$$g(u) = \frac{k[(u/\tanh u) - (u_0/\tanh u_0)]^{\frac{1}{2}}(\operatorname{sech}^2 u + \tanh u/u)}{(\tanh u_0/u_0) - (\tanh u/u) + \tanh^2 u - \tanh^2 u_0 + u_0^2 k^4 + k^2 - 1}$$

and fixing k, one finds that $g(u)$ is increasing on an interval $[0, u_m]$ $u_m > 0$, attains a maximum $g(u_m) = g_m$ and decreases on $[u_m, \infty)$ with $\lim g(u) = 0$ as $u \to \infty$. Moreover, as $k \to 1_-$, one has $u_0 \to 0$ and the value of g_m increases without bound. Thus for each fixed k with $0 < k < 1$ and each θ such that $0 < \tan\theta < g_m$, (3.1) yields two distinct stationary points $u = u_t, u_d$ with $u_0 < u_t < u_m < u_d < \infty$. As $\tan\theta \to g_m$ the stationary points coalescence and for $\tan\theta = 0$ one has only the single stationary point $u = u_t = u_0$. Therefore, taking into account Theorem 3, the pattern exists in the downstream sector $0 \leqq |\theta| \leqq \theta_m^k < \pi/2$, where $\theta_m^k = \tan^{-1} g_m$, and as $k \to 1_-$, the sector widens out to the half plane $x > 0$.

As for the asymptotic wave amplitude: within the sector, that is, for fixed θ with $0 \leqq |\theta| < \theta_m^k$ the distinct stationary points u_t and u_d each yield a contribution of order $O(r^{-\frac{1}{2}})$. As θ varies, these contributions generate two distinct wave systems whose level curves are given by the parametric equations

$$rp_-(u)\cos\theta = \text{constant}, \qquad \tan\theta = g(u)$$

where $u_0 \leqq u < u_m$ gives a transverse system and $u_m < u < \infty$ a diverging system. On the border of the sector $|\theta| = \theta_m^k$ the stationary points u_t, u_d coalesce to a single stationary point $u = u_m$ of second order yielding a contribution of order $O(r^{-\frac{1}{3}})$. Thus the pattern resembles the pattern obtained by Havelock in the case of a pressure point moving on water of finite depth with subcritical velocity [4].

In the neighborhood of the border: $\theta_m^k - \delta \leqq |\theta| \leqq \theta_m^k$ the ordinary method of stationary phase does not yield an expansion uniform in θ due to the proximity of the stationary points. Instead, we use the method of Chester *et al.*, as described in Sirovich [9]. The idea is to introduce a change of variables $u = u(t)$ so that the phase function goes over into a special cubic form in t

$$k^{-1}p_-(u(t))\cos\theta = -t^3/3 + a^2 t + b$$

where the parameter a characterizes the proximity of the stationary points. It has been shown [6] that the transformation $u = u(t)$ will be 1–1 on the interval of integration, that is, $u'(t) \neq 0$, if a and b are chosen so that $t = -a, a$ correspond to $u = u_t, u_d$ respectively, which leads to

$$a^3 = \tfrac{3}{4}k^{-1}\cos\theta[p_-(u_d) - p_-(u_t)],$$
$$b = \tfrac{1}{2}k^{-1}\cos\theta[p_-(u_d) + p_-(u_t)].$$

The expression (2.5) then becomes, after neglecting the second integral which is of order $O(r^{-1})$, setting $t_0 = -t(u_0) > 0$ and restricting θ to $0 \leqq \theta < \pi/2$,

$$\eta(r, \theta) = \mathrm{Im} \int_{\infty}^{-t_0} \frac{u(t)u'(t)\, e^{ir(-t^3/3 + a^2 t + b)} G_-(u(t))}{\cosh u(t)[(k^2 u(t)/\tanh u(t)) - 1]^{\frac{1}{2}}}\, dt. \tag{4.1}$$

One obtains the uniform expansion of this integral by expanding the coefficient of the exponential function in the form

$$\frac{u(t)u'(t)G_-(u(t))}{\cosh u(t)[(k^2 u(t)/\tanh u(t)) - 1]^{\frac{1}{2}}} = c_0 + c_1 t + (t^2 - a^2)H(t)$$

and integrating term by term. We obtain only the leading term of the expansion which can be shown to be the term of lowest order [9]. In this case the leading term is the first term since substituting $t = \pm a$ in the above one solves for c_0

$$c_0 = \frac{1}{2k}\left\{\frac{u_t u'(-a)G_-(u_t)}{\cosh u_t[(k^2 u_t/\tanh u_t) - 1]^{\frac{1}{2}}} + \frac{u_d u'(a)G_-(u_d)}{\cosh u_d[(k^2 u_d/\tanh u_d) - 1]^{\frac{1}{2}}}\right\}$$

which is non-zero since $u'(t)$ is non-zero on the interval of integration. Thus the leading term of the uniform expansion of (4.1) is

$$\eta(r, \theta) = c_0\, \mathrm{Im} \int_{\infty}^{-t_0} e^{ir(-t^3/3 + a^2 t + b)}\, dt. \tag{4.2}$$

This can be expressed in terms of the Airy function [2]

$$Ai(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-q^3/3 + zq}\, dq$$

which is real for z real. One merely extends the path of integration in (4.2) from $-t_0$ to $-\infty$, introducing an error of $O(r^{-1})$; then, performing the change of variable $q = -ir^{\frac{1}{3}}t$, one obtains

$$\eta(r, \theta) = -2\pi c_0 r^{-\frac{1}{3}} \sin(br) Ai(-a^2 r^{\frac{2}{3}}),$$

valid in a finite neighborhood of the border: $\theta_m^k - \delta \leqq |\theta| \leqq \theta_m^k$. We remark that the next term in the expansion of (4.1) will be of order $O(r^{-\frac{2}{3}})$ and will involve the derivative of the Airy function.

For large depth, $k \ll 1$, the pattern closely resembles the ship-wave pattern for infinite depth except reduced in amplitude. To show this we note that u_0 will be large. Hence, taking $\tanh \simeq 1$ for $u \geqq u_0$, letting $v = (k^2 u - 1)^{\frac{1}{2}}$ and $v = -(k^2 u - 1)^{\frac{1}{2}}$ in the first and second integrals respectively, introducing the parameter $N = rk^{-2}$, which is independent of h, and taking $\cos\theta > 0$, (2.5) becomes

$$\eta(r, \theta) = -2\, \mathrm{Im} \int_{-\infty}^{\infty} \frac{(1 + v^2)\, e^{iN(\cos\theta - v\sin\theta)(1 + v^2)^{\frac{1}{2}}}}{k^4 \cosh[k^{-2}(1 + v^2)]}\, dv. \tag{4.3}$$

We compare (4.3) with the integral

$$\eta(r, \theta) = -C \,\mathrm{Im} \int_{-\infty e^{i\pi/8}}^{\infty e^{-i\pi/8}} (1 + v^2)\, e^{iN(\cos\theta - v \sin\theta)(1+v^2)^{\frac{1}{2}}}\, dv \tag{4.4}$$

where $C > 0$, which yields the wave amplitude of the ship-wave pattern for infinite depth, Ursell [7]. One sees that (4.3), which has the same stationary points as (4.4), must yield the same wave pattern except attenuated in amplitude due to the presence of the denominator. In fact as $k \to 0$ one has $\eta(r, \theta) \to 0$.

Using the method of Chester *et al.*, Ursell [7], has obtained a uniform expansion of (4.4) near the border $|\theta| = \cot^{-1} 2\sqrt{2}$ of the wave sector. One can do this in a quite similar way for (4.3). The result is, of course, a special case of our previous result but is given by a more explicit expression. In the previous notation one gets, setting $Q = (1 - 8\tan^2\theta)^{\frac{1}{2}}$,

$$a^3 = \frac{3^{\frac{5}{2}}}{64} \frac{\cos^2\theta}{\sin\theta} [(1 - Q/3)^{\frac{3}{2}}(1 + Q)^{\frac{1}{2}} - (1 + Q/3)^{\frac{3}{2}}(1 - Q)^{\frac{1}{2}}],$$

$$b = \frac{3^{\frac{5}{2}}}{32} \frac{\cos^2\theta}{\sin\theta} [(1 - Q/3)^{\frac{3}{2}}(1 + Q)^{\frac{1}{2}} + (1 + Q/3)^{\frac{3}{2}}(1 - Q)^{\frac{1}{2}}],$$

$$c_0 = \frac{3^{\frac{5}{4}} \cos^2\theta}{2^{\frac{11}{2}} k^4 \sin^{\frac{5}{2}}\theta} \left(\frac{a}{Q}\right)^{\frac{1}{2}} \left\{ \frac{(1 + Q)^{\frac{5}{4}}(1 - Q/3)^{\frac{5}{4}}}{\cosh\{k^{-2}[1 + \frac{1}{2}(1 + Q)(1 - Q)^{-1}]\}} + \frac{(1 - Q)^{\frac{5}{4}}(1 + Q/3)^{\frac{5}{4}}}{\cosh\{k^{-2}[1 + \frac{1}{2}(1 - Q)(1 + Q)^{-1}]\}} \right\},$$

and the leading term of the uniform expansion is

$$\eta(r, \theta) = -2\pi c_0 N^{-\frac{1}{3}} \sin(bN) Ai(-a^2 N^{\frac{2}{3}}) + O(N^{-\frac{2}{3}})$$

valid for θ in a finite neighborhood $\cot^{-1} 2\sqrt{2} - \delta \leqq |\theta| \leqq \cot^{-1} 2\sqrt{2}$.

5. Flow at critical speed

We now discuss the steady state flow for large r at the critical speed $k = 1$. We recall that its existence has been demonstrated in Theorem 1. We find that the wave pattern has one component (diverging) existing in the region $0 < |\theta| < \pi/2$ downstream from the obstacle. In the region $0 < |\theta| \leqq \pi/2 - \delta$ the wave amplitude is shown to be of order $O(r^{-\frac{1}{2}})$ using the ordinary method of stationary phase. Near the transverse line $x = 0$, that is, for $\pi/2 - \delta \leqq |\theta| < \pi/2$, the ordinary method of stationary phase does not yield an expansion uniform in θ so we use a method of Bleistein [8], obtaining a uniform expansion in terms of Fresnel integrals.

We investigate the distribution of stationary points of $p_-(u)$ as θ varies. In the present case, $g(u)$ in (3.1) can be written in the form

$$g(u) = \frac{[(u/\tanh u) - 1]^{\frac{1}{2}}[1 + u/(\cosh u \sinh u)]}{[(u/\tanh u) - 1] + u \tanh u}$$

and one finds that $g(u)$ is positive and decreasing on $(0, \infty)$ with $\lim_{u\to 0_+} g(u) = \infty$ and $\lim_{u\to\infty} g(u) = 0$. Thus for each value of $\tan\theta > 0$, (3.1) has a single root $u = u_d$, $0 < u_d < \infty$ which by Theorem 3 contributes in the downstream direction $0 < \theta < \pi/2$, and moreover as $\tan\theta \to \infty$ one has $u_d \to 0$. In the transverse direction, $\cos\theta = 0$, and (3.2) is not valid but then one can verify directly that $u = 0$ is a stationary point of the phase functions $p_\pm(u)\cos\theta$. However, since $G_\pm(0) = 0$, this stationary point does not contribute to the wave pattern.

As for the asymptotic wave amplitude: within the right half plane, that is, for fixed θ with $0 < |\theta| < \pi/2$ the stationary point $u = u_d$ yields a contribution of order $O(r^{-\frac{1}{2}})$. As θ varies, this contribution generates a diverging wave system whose level curves are given by

$$\left.\begin{aligned} rp_-(u)\cos\theta &= \text{constant} \\ \tan\theta &= g(u) \end{aligned}\right\} 0 < u < \infty.$$

Near the transverse line $x = 0$, that is for θ with $\pi/2 - \delta \leqq |\theta| < \pi/2$, the ordinary method of stationary phase does not yield an expansion uniform in θ due to the proximity of the stationary point and the endpoint of the interval of integration $u = 0$. Instead, we use a method of Bleistein [8] to obtain a uniform expansion.

We first write $q_\pm(u) = p_\pm(u)\cot\theta$. Clearly, $q_-(u)$ has the stationary point $u = u_d$ and as $\cot\theta \to 0_+$, u_d approaches the endpoint $u = 0$ of the first integral in (2.5). Writing the phase functions and the integrands of (2.5) in the form

$$q_\pm(u) = u(\tanh u/u)^{\frac{1}{2}}\cot\theta \pm u^2\left[\frac{u - \tanh u}{u^3}\right]^{\frac{1}{2}},$$

$$J_\pm(u) = \frac{G_\pm(u)}{\cosh u}\left[\frac{u^2\tanh u}{u - \tanh u}\right]^{\frac{1}{2}},$$

one sees that they can be defined for $u < 0$ by the formulas

$$q_\pm(-u) = -q_\mp(u), \qquad J_\pm(-u) = J_\mp(u)$$

and will be analytic in a neighborhood of the entire real axis. Then, going over from u to $-u$ in the second integral, (2.5) becomes

$$\eta(r\,\theta) = \operatorname{Im}\left\{\int_\infty^0 J_-(u)\,e^{ir\,q_-(u)\sin\theta}du - \int_{-\infty}^0 J_-(u)\,e^{-ir\,q_-(u)\sin\theta}du\right\}. \tag{5.1}$$

Following Bleistein, we introduce a change of variable $u = u(t)$ so that $q_-(u)$ goes over into a special quadratic form:

$$q_-(u(t))\sin\theta = -t^2/2 + at, \qquad a > 0 \tag{5.2}$$

where the parameter a characterizes the proximity of the stationary point and the endpoint. It has been shown [8] that the transformation $u = u(t)$ will be 1–1 from $-\infty < t < \infty$ to $-\infty < u < \infty$, i.e. $u'(t) \neq 0$, if a is chosen so that $u = u_d$ corresponds to $t = a$ in $u(t)$ which leads to

$$a = (2q_-(u_d)\sin\theta)^{\frac{1}{2}}.$$

One notes that $a = a(\theta) \to 0$ as $u_d \to 0$, i.e. as $\theta \to \pi/2_-$. (5.1) then becomes

$$\eta(r, \theta) = \operatorname{Im}\left\{\int_{\infty}^{0} J_-(u(t))u'(t)\, e^{-ir(t^2/2-at)}\, dt - \int_{-\infty}^{0} J_-(u(t))u'(t)\, e^{ir(t^2/2-at)}\, dt\right\}. \tag{5.3}$$

We expand the coefficient of the exponential functions in the form

$$J_-(u(t))u'(t) = c_0 + c_1 t + t(t-a)H(t).$$

We obtain only the leading term of the expansion which can be shown to be the term of lowest order. Substituting $t = 0$ one obtains

$$c_0 = J_-(u(0))u'(0) = J_-(0)u'(0).$$

Now $G_-(0)$ (c.f. (2.5)) is equal to 1 as long as $|\theta| < \pi/2$. Differentiating (5.2) with respect to t and using L'Hospital's rule, one gets

$$[u'(a)]^2 = \frac{-1}{q''_-(u_d)\sin\theta}$$

and for $a = u_d = 0$

$$[u'(0)]^2 = \frac{-1}{q''_-(0)\sin\theta} = \frac{3}{2\sin\theta}.$$

Then $c_0 = 3(2\sin\theta)^{-\frac{1}{2}}$, $0 < \theta < \pi/2$. Thus, going over from t to $-t$ in the second integral, the leading term of the uniform expansion of (5.3) is

$$\eta(r, \theta) = -c_0 \operatorname{Im}\left\{\int_{0}^{\infty} e^{ir(t^2/2+at)}\, dt + \int_{0}^{\infty} e^{-ir(t^2/2-at)}\, dt\right\}. \tag{5.4}$$

This can be expressed in terms of the Fresnel integrals

$$S(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\sqrt{x}} \sin q^2\, dq, \qquad C(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\sqrt{x}} \cos q^2\, dq$$

for, setting $p = (r/2)^{\frac{1}{2}} t$ in (5.4) and completing the square in the exponent, one finally gets

$$\eta(r, \theta) = 2\sqrt{\pi}\, c_0 r^{-\frac{1}{2}}[\cos(a^2 r/2)S(a^2 r/2) - \sin(a^2 r/2)C(a^2 r/2)]$$

valid in a finite neighborhood $\pi/2 - \delta \leqq \theta < \pi/2$.

Using this expansion one can evaluate the limiting wave amplitude as the line transverse to the flow direction, $\theta = \pi/2$, is approached for fixed r. In fact, one sees that as $\theta \to \pi/2_-$, i.e. as $a \to 0$, one has $\eta(r, \theta) \to 0$.

6. Flow at supercritical speed

We now discuss the steady state wave pattern (2.5) for supercritical speed $k > 1$. We find that the pattern has one component (diverging) existing in the region $0 < |\theta| < \theta_m^k < \pi/2$ where $\theta_m^k = \tan^{-1}[(k^2 - 1)^{-\frac{1}{2}}]$. In the region $0 < |\theta| < \theta_m^k - \delta$ the wave amplitude is of order $O(r^{-\frac{1}{2}})$. In the region $\theta_m^k - \delta \leqq |\theta| < \theta_m^k$, we use the method of Chester *et al.* to obtain a uniform expansion in terms of the Airy function and we find that in the limit as $|\theta| \to \theta_m^k$ the wave amplitude is of order $O(r^{-\frac{1}{3}})$.

We investigate the distribution of stationary points of $p_-(u)$ as θ varies. In the present case, $g(u)$ in (3.1) can be written in the form

$$g(u) = \frac{[(k^2u/\tanh u) - 1]^{\frac{1}{2}}(\operatorname{sech} u + \tanh u/u)}{k^2 - (\tanh u/u) + \tanh^2 u + k^2 - 1}$$

and one finds that $g(u)$ is positive and decreasing on $[0, \infty)$ with $g(0) = (k^2 - 1)^{-\frac{1}{2}}$ and $\lim_{u\to\infty} g(u) = 0$. Thus for each value of $\tan\theta$ such that $0 < \tan\theta < (k^2 - 1)^{-\frac{1}{2}}$, (3.1) has a single root $u = u_d$ which by Theorem 3 contributes in the downstream direction $0 < \theta < \theta_m^k$ where $\theta_m^k = \tan^{-1}[(k^2 - 1)^{-\frac{1}{2}}]$ and, moreover, $u_d \to 0$ as $\tan\theta \to (k^2 - 1)^{-\frac{1}{2}}$. For $\tan\theta = (k^2 - 1)^{-\frac{1}{2}}$ one can verify directly that $p_-(u)$ has a stationary point at $u = 0$, but since $G_-(0) = 0$, it doesn't contribute to the wave pattern.

As for the asymptotic wave amplitude: for fixed θ with $0 < |\theta| < \theta_m^k$, the stationary point $u = u_d$ yields a contribution of order $O(r^{-\frac{1}{2}})$. As θ varies this contribution generates a diverging wave system whose level curves are given by

$$\left.\begin{aligned} r\, p_-(u)\cos\theta &= \text{constant} \\ \tan\theta &= g(u) \end{aligned}\right\} 0 < u < \infty.$$

In the region $\theta_m^k - \delta \leqq |\theta| < \theta_m^k$, the stationary point $u = u_d$ is near the endpoint of the interval of integration of the first integral $u = 0$, and the ordinary method of stationary phase does not yield an expansion uniform in θ. In this case, however, the method of Bleistein is not suitable. This becomes evident when one writes the function $p_-(u)$ in the form

$$p_-(u) = u(\tanh u/u)^{\frac{1}{2}} - u[k^2 - (\tanh u/u)]^{\frac{1}{2}} \tan\theta$$

and extends the domain of definition to $u < 0$. One then has

$$p_-(-u) = -p_-(u) \tag{6.1}$$

so that, in addition to $u = u_d$, $p_-(u)$ has a stationary point at $u = -u_d$, outside the interval of integration, and the two coalesce to the endpoint $u = 0$ as $\tan\theta \to (k^2 - 1)^{-\frac{1}{2}}$. Thus one actually has a case of two coalescing stationary points; the point to which they coalesce just happens to be an endpoint. Hence it is appropriate to use the method for Chester, *et al.*

We choose a transformation $u = u(t)$ so that the phase function goes over to a special cubic form.

$$k^{-1} p_-(u(t))\cos\theta = -t^3/3 + a^2t + b \tag{6.2}$$

where the parameter a characterizes the proximity of the stationary points. The transformation will be 1–1 for $-\infty < t < \infty$, i.e. $u'(t) \neq 0$, if a and b are chosen so that $t = -a, a$ correspond to $u = -u_d, u_d$ respectively, which leads to, in view of (6.1),

$$a^3 = \tfrac{3}{4}k^{-1}\cos\theta[p_-(u_d) - p_-(-u_d)] = \tfrac{3}{2}k^{-1} p_-(u_d)\cos\theta,$$

$$b = \tfrac{1}{2}k^{-1}\cos\theta[p_-(u_d) + p_-(-u_d)] = 0.$$

The expression (2.5) then becomes, neglecting the second integral which is of order $O(r^{-1})$, and restricting θ to $(0, \theta_m^k)$

$$\eta(r, \theta) = \operatorname{Im} \int_{\infty}^{0} \frac{u(t)u'(t)\, e^{ir(-t^3/3+a^2t)} G_-(u(t))}{\cosh u(t)[(k^2u(t)/\tanh u(t)) - 1]^{\frac{1}{2}}}\, dt. \tag{6.3}$$

We expand the coefficient of the exponential function in the form

$$J(t) = \frac{u(t)u'(t)G_-(u(t))}{\cosh u(t)[(k^2u(t)/\tanh u(t)) - 1]^{\frac{1}{2}}} = c_0 + c_1 t + (t^2 - a^2)H(t)$$

and integrate term by term. We obtain only the leading term of the expansion which can be shown to be the term of lowest order. In the present case, the leading term will be the *second* term, since substituting $t = \pm a$ in the above, one solves for c_0 and c_1

$$c_0 = \tfrac{1}{2}[J(a) + J(-a)], \qquad c_1 = \frac{1}{2a}[J(a) - J(-a)].$$

Now differentiating (6.2) with respect to t and using L'Hospital's rule one gets

$$[u'(\pm a)]^2 = \frac{\mp 2a}{p_{-uu}(\pm u_a)k^{-1}\cos\theta}$$

and, in view of (6.1), one has $u'(a) = u'(-a)$ and therefore $J(a) = -J(-a)$, whence $c_0 = 0$ and $c_1 = J(a)/a$. Now c_1 is non-zero for all $a > 0$, and, expanding $u(t)$ about $t = 0$, $u_a = u'(0)a + O(a^2)$ from which one obtains

$$\lim_{a\to 0+} c_1 = [u'(0)]^2(k^2 - 1)^{-\frac{1}{2}} \neq 0.$$

Thus the leading term of the uniform expansion of (6.3) will be

$$\eta(r, \theta) = c_1 \operatorname{Im} \int_{\infty}^{0} t\, e^{ir(-t^3/3+a^2t)}\, dt.$$

This can be expressed in terms of the derivative of the Airy function defined in Section 4. One first expresses it as an integral from $-\infty$ to ∞ since the integrand (including Im) is even, and then, letting $q = -ir^{\frac{1}{3}}t$ one finally gets

$$\eta(r, \theta) = -c_1\pi r^{-\frac{2}{3}} \operatorname{Re} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} q\, e^{(-q^3/3 - a^2 r^{\frac{2}{3}} q)}\, dq = -c_1\pi r^{-\frac{2}{3}} Ai'(-a^2 r^{\frac{2}{3}}).$$

From this one deduces that $\lim_{|\theta|\to\theta_m^k} \eta(r, \theta) = \lim_{a\to 0+} \eta(r, \theta) = -c_1\pi r^{-\frac{2}{3}} Ai'(0)$.

Appendix A

Proof of Theorem 1. Letting $\bar{\gamma} = 1$, replacing φ by $-\varphi$ for $-\pi \leqq \varphi \leqq 0$, and setting $T = t(g/h)^{\frac{1}{2}}$ and

$$p_{\pm}(u, \varphi) = ku\cos\varphi \pm (u\tanh u)^{\frac{1}{2}}, \tag{A.1}$$

the time dependent terms in (1.15) become

$$\eta^t(r,\theta) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{\infty} \frac{u^2(\tanh u/u)^{\frac{1}{2}}}{\cosh u} \times \left\{ \frac{e^{-iTp_+(u,\varphi)}}{p_+(u,\varphi)} - \frac{e^{-iTp_-(u,\varphi)}}{p_-(u,\varphi)} \right\} [e^{iur\cos(\varphi-\theta)} + e^{iur\cos(\varphi+\theta)}]\,du\,d\varphi \tag{A.2}$$

where by hypothesis the path of u-integration is deformed below (above) $u = u(\varphi)$ when $0 \leqq \varphi \leqq \pi/2$ ($\pi/2 < \varphi \leqq \pi$). We shall estimate the double integral (A.2) as $T \to \infty$ by obtaining estimates on the u-integral which are uniform in φ for φ in the interval of integration. These same estimates will then apply to the double integral.

We wish to bring the integrals through the curly brackets. From (2.1) and (A.1), $u(\varphi)$ is defined by

$$u^{-2}p_+(u,\varphi)p_-(u,\varphi) = 0.$$

For $0 \leqq \varphi \leqq \pi/2$ one has $u^{-1}p_+(u,\varphi) > 0$ for all $u > 0$ so that $u(\varphi)$ is defined by $u^{-1}p_-(u,\varphi) = 0$, i.e.

$$k\cos\varphi = (\tanh u/u)^{\frac{1}{2}} \tag{A.3}$$

and only the second term in the curly brackets can have a pole. Thus, for $0 \leqq \varphi \leqq \pi/2$, one can bring the integrals through the curly brackets and return to a straight path through $u = u(\varphi)$ in the u-integral of the first term. Now since $p_+(u,\varphi)$ is strictly increasing, $u \geqq 0$, $0 \leqq \varphi \leqq \pi/2$, it has no stationary points and so, by the Riemann–Lebesgue Lemma, the u-integral of the first term is $O(T^{-1})$ for $0 \leqq \varphi \leqq \pi/2$. Similarly, the u-integral of the second term will be $O(T^{-1})$ for $\pi/2 < \varphi \leqq \pi$ and one obtains, neglecting the terms of $O(T^{-1})$ and going over from φ to $\pi - \varphi$ in the first term

$$\eta^t(r,\theta) = -\frac{1}{4\pi}\int_0^{\pi/2}\left\{\int_0^{\infty} \frac{u^2(\tanh u/u)^{\frac{1}{2}}}{\cosh u}\,\frac{e^{iTp_-(u,\varphi)}}{p_-(u,\varphi)}[e^{-iur\cos(\varphi-\theta)} + e^{-iur\cos(\varphi+\theta)}]\,du + \int_0^{\infty} \frac{u^2(\tanh u/u)^{\frac{1}{2}}}{\cosh u}\,\frac{e^{-iTp_-(u,\varphi)}}{p_-(u,\varphi)}[e^{iur\cos(\varphi-\theta)} + e^{iur\cos(\varphi+\theta)}]\,du\right\}d\varphi \tag{A.4}$$

where the path of u-integration is deformed into a semi-circle of radius ε above (below) $u = u(\varphi)$ in the first (second) integral.

We now determine the stationary points of $p_-(u,\varphi)$ for fixed φ, $0 \leqq \varphi \leqq \pi/2$. Differentiating $p_-(u,\varphi)$ with respect to u, one finds that p_- has a stationary point $u = u_s$ defined implicitly by the equation

$$k\cos\varphi = (\tanh u_s/u_s)^{\frac{1}{2}}\left[\frac{1 + 2u_s/\sinh 2u_s}{2}\right]. \tag{A.5}$$

We will need to know the location of u_s with respect to $u(\varphi)$. From (A.3) and (A.5) one has

$$[\tanh u(\varphi)/u(\varphi)]^{\frac{1}{2}} = (\tanh u_s/u_s)^{\frac{1}{2}}\left[\frac{1 + 2u_s/\sinh 2u_s}{2}\right] \leqq (\tanh u_s/u_s)^{\frac{1}{2}}$$

where equality holds iff $u_s = 0 = u(\varphi)$. From this one has the following alternative:

$$\begin{aligned} &\text{a)}\ u(\varphi) = u_s = 0 \qquad (\text{iff } k\cos\varphi = 1) \\ &\text{b)}\ 0 < u_s < u(\varphi) \qquad (\text{iff } 0 < k\cos\varphi < 1) \end{aligned} \tag{A.6}$$

and, moreover, $\lim_{\varphi\to\cos^{-1}(k^{-1})_+} u(\varphi) = \lim_{\varphi\to\cos^{-1}(k^{-1})_+} u_s = 0$.

We now obtain estimates on the u-integrals in (A.4) which are uniform in φ. We first take the case $k < 1$. Then, one has $0 < u_0 \leqq u(\varphi)$ for $0 \leqq \varphi \leqq \pi/2$; hence, in view of (A.6), the pole $u(\varphi)$ and the stationary points u_s are bounded away from the origin and from each other. Thus one can choose the radius ε small enough that $0 < u_s < u(\varphi) - \varepsilon$, $0 \leqq \varphi \leqq \pi/2$, i.e. so that u_s lies on the straight section of the path between the origin and the semicircle. Then the straight section $[0, u(\varphi) - \varepsilon]$ of both u-integrals yields a contribution of order $O(T^{-\frac{1}{2}})$ due to the stationary point u_s, and the straight section $[u(\varphi) + \varepsilon, \infty)$ yields $O(T^{-1})$ from the Riemann–Lebesgue Lemma. Both of these contributions are uniform in φ, $0 \leqq \varphi \leqq \pi/2$.

For u on the semicircle, we expand $p_-(u, \varphi)$ about $u = u(\varphi)$. For the first u-integral in (A.4) one obtains

$$\begin{aligned} p_-(u,\varphi) &= \frac{u(\varphi)}{2}\,[\tanh u(\varphi)/u(\varphi)]^{-\frac{1}{2}}\,|(\tanh u/u')|_{u=u(\varphi)}\,\varepsilon\,e^{i\alpha_+} + O(\varepsilon^2) \\ &= C\varepsilon\, e^{i\alpha_+} + O(\varepsilon^2), \qquad C > 0,\ \pi > \alpha_+ > 0 \end{aligned}$$

and in the second integral

$$p_-(u,\varphi) = C\varepsilon\, e^{i\alpha_-} + O(\varepsilon^2), \qquad -\pi < \alpha_- < 0.$$

Then the time dependent exponents become

$$\begin{aligned} &\text{a)}\quad iTp_-(u,\varphi) = CT\varepsilon\, e^{i(\alpha_+ + \pi/2)} + TO(\varepsilon^2), \\ &\text{b)}\ -iTp_-(u,\varphi) = CT\varepsilon\, e^{i(\alpha_- - \pi/2)} + TO(\varepsilon^2), \end{aligned} \tag{A.8}$$

for the first and second u-integrals respectively. Both exponents have negative real part for ε sufficiently small. Therefore the integrand of both u-integrals tends to zero exponentially on the semicircle as $T \to \infty$. Hence for $k < 1$ the u-integrals are of order $O(T^{-\frac{1}{2}})$ uniformly in φ, $0 \leqq \varphi \leqq \pi/2$. Therefore the double integral (A.4) is also of order $O(T^{-\frac{1}{2}})$.

We now consider the case $k \geqq 1$. Then one has $0 \leqq u(\varphi) < \infty$ for $\cos\varphi \leqq k^{-1}$ and moreover $u(\varphi) \to 0$ as $\cos\varphi \to k^{-1}{}_-$. For $k^{-1} < \cos\varphi \leqq 1$, $p_-(u, \varphi)$ has no stationary point nor does $u^{-1}p_-(u, \varphi)$ have a zero. Hence for these values of φ one can return to a straight path in the u-integral and thus the double integral for $0 \leqq u < \infty$, $k^{-1} < \cos\varphi \leqq 1$, is $O(T^{-1})$ by the Riemann–Lebesgue Lemma. Now for $\cos\varphi \leqq k^{-1} - \delta$, $\delta > 0$, the pole $u(\varphi)$ and stationary point u_s are bounded away from the origin and from each other, c.f. (A.6), and one proceeds as in the previous case, $k < 1$, and finds that the u-integral is $O(T^{-\frac{1}{2}})$ uniformly in φ. Thus the double integral for $0 \leqq u < \infty$, $\cos\varphi \leqq k^{-1} - \delta$, is of order $O(T^{-\frac{1}{2}})$.

The main difficulty occurs when $k^{-1} \geqq \cos\varphi \geqq k^{-1} - \delta$ for then $u(\varphi)$ and u_s coalesce to the origin as $\varphi \to \cos^{-1}(k^{-1})_+$, c.f. (A.6). For this case we first take u on the semicircle, expanding $p_-(u, \varphi)$ about $u(\varphi)$ as in (A.8a, b). Our previous choice of constant ε

as φ varies is no longer suitable. In fact, since the center of the semicircle, $u = u(\varphi)$, tends to the origin as $\varphi \to \cos^{-1}(k^{-1})_+$ and the path of u-integration terminates at the origin, the radius ε of the semicircle will have to diminish of order $O[u(\varphi)]$ as $\varphi \to \cos^{-1}(k^{-1})_+$ in order that the semicircle remain on the path. Accordingly, we make the following judicious choice of ε: $\varepsilon = T^{-\frac{1}{2}}u(\varphi)$, $T > 1$ and the exponents become, c.f. (A.8a, b),

$$\pm \mathrm{i}Tp_-(u, \varphi) = C_1 T^{\frac{1}{2}} \mathrm{e}^{i(\alpha_\pm \pm \pi/2)} + O(T^0), \qquad C_1 > 0,$$

where as before $\pi > \alpha_+ > 0$, $-\pi < \alpha_- < 0$. Now the first term has negative real part and the remaining terms do not grow large with T and so the exponential functions will go to zero on the semicircle as $T \to \infty$, uniformly in φ, $k^{-1} \geqq \cos\varphi \geqq k^{-1} - \delta$. Thus the double integral for u on the semicircle and $k^{-1} \geqq \cos\varphi \geqq k^{-1} - \delta$ goes to zero exponentially as $T \to \infty$.

We now take u on the straight section $[0, u(\varphi)(1 - T^{-\frac{1}{2}})]$. We expand (A.3) and (A.5) for small $u(\varphi)$ and u_s getting $u_s = u(\varphi)/3$. Thus for sufficiently large T, i.e. $T > \frac{9}{4}$, the stationary point u_s will be located on the above straight section, and moreover coalesces to the endpoint $u = 0$ as $\varphi \to \cos^{-1}(k^{-1})_+$. To obtain an estimate, asymptotic in T and uniform in φ, $k^{-1} \geqq \cos\varphi \geqq k^{-1} - \delta$, of the u-integral on the above straight section we turn to the uniform asymptotic method of Chester, *et al.* [6], c.f. Sec. 4. Writing the phase function in the form

$$p_-(u, \varphi) = u[k \cos\varphi - (\tanh u/u)^{\frac{1}{2}}]$$

one sees that it can be defined as an analytic function in a neighborhood of the entire real axis by the formula

$$p_-(-u, \varphi) = -p_-(u, \varphi). \tag{A.9}$$

Differentiating (A.9), it is evident that $p_-(u)$ has a stationary point at $u = -u_s$, outside the interval of integration, in addition to that at $u = u_s$ and the two coalesce to $u = 0$ as $\varphi \to \cos^{-1}(k^{-1})_+$. We introduce the transformation $u = u(t)$ defined by

$$p_-(u(t), \varphi) = t^3/3 - a^2 t + b \tag{A.10}$$

where $a = a(\varphi)$, $b = b(\varphi)$, and require that $t = -a, a$ correspond to $u = -u_s, u_s$ in $u(t)$ which leads to

$$a^3 = \tfrac{3}{4}[p_-(-u_s, \varphi) - p_-(u_s, \varphi)],$$

$$b = \tfrac{1}{2}[p_-(-u_s, \varphi) + p_-(u_s, \varphi)],$$

or, in view of (A.9),

$$a^3 = 3p_-(-u_s, \varphi)/2, \qquad b = 0.$$

Substituting into (A.4) for the values of φ and u we are considering, denoting

$$\varphi_0 = \cos^{-1}(k^{-1} - \delta), \ \varphi_1 = \cos^{-1}(k^{-1}),$$

$$I = -\frac{1}{4\pi}\int_{\varphi_0}^{\varphi_1}\left\{\int_0^{t_\varphi}\frac{u^2(t)[\tanh u(t)/u(t)]^{\frac{1}{2}}u'(t)}{\cosh u(t)}\frac{e^{iT(t^3/3-a^2t)}}{t^3/3-a^2t}\right.$$

$$\cdot[e^{-iu(t)r\cos(\varphi-\theta)}+e^{-iu(t)r\cos(\varphi+\theta)}]dt$$

$$+\int_0^{t_\varphi}\frac{u^2(t)[\tanh u(t)/u(t)]^{\frac{1}{2}}u'(t)}{\cosh u(t)}\frac{e^{-iT(t^3/3-a^2t)}}{t^3/3-a^2t}$$

$$\left.\cdot[e^{iu(t)r\cos(\varphi-\theta)}+e^{iu(t)r\cos(\varphi+\theta)}]dt\right\}d\varphi \qquad (A.11)$$

where $u(t_\varphi) = u(\varphi)(1 - T^{-\frac{1}{3}})$. We expand the coefficients of the exponential functions of T in the form

$$J_\pm(t) = \frac{u^2(t)[\tanh u(t)/u(t)]^{\frac{1}{2}}u'(t)}{\cosh u(t)[t^3/3 - a^2t]}[e^{\pm iu(t)r\cos(\varphi-\theta)} + e^{\pm iu(t)r\cos(\varphi+\theta)}]$$

$$= c_{0\pm} + c_{1\pm}t + (t^2 - a^2)H_\pm(t)$$

where $c_{0\pm} = c_{0\pm}(\varphi)$, $c_{1\pm} = c_{1\pm}(\varphi)$, and $H_\pm(t) = H_\pm(t, \varphi)$. The leading term can be shown to yield the term of lowest order in the expansion of (A.11). Substituting $t = \pm a$, one solves for c_0

$$c_{0\pm} = \tfrac{1}{2}[J_\pm(a) + J_\pm(-a)].$$

Differentiating (A.10) with respect to t and using L'Hospital's rule

$$[u'(\pm a)]^2 = 2a/p''_-(\pm u_s), \qquad a \neq 0$$

and since $p''_-(-u_s) = -p''_-(u_s)$ from (A.9), one has $u'(-a) = u'(a)$. Another application of L'Hospital's rule yields an expression for $u'(\pm a)$ valid also for $a = 0$

$$[u'(\pm a)]^3 = 2/p'''(\pm u_s) = 2/p'''(u_s)$$

where we have again used (A.9). One then gets

$$c_{0+} = -\frac{3iu_s^2(\tanh u_s/u_s)^{\frac{1}{2}}u'(a)}{2a^3\cosh u_s}[\sin(ru_s\cos(\varphi-\theta)) + \sin(ru_s\cos(\varphi+\theta))]$$

and $c_{0-} = -c_{0+} = -c_0$.

The leading term of (A.11) is then

$$I = (2\pi)^{-1}\int_{\varphi_0}^{\varphi_1} ic_0(\varphi)\int_0^{t_\varphi}\sin[T(t^3/3 - a^2(\varphi)t)]dt\,d\varphi$$

or, letting $q = T^{\frac{1}{3}}t$,

$$I = (T^{-\frac{1}{3}}/2\pi)\int_{\varphi_0}^{\varphi_1} ic_0(\varphi)\int_0^{T^{\frac{1}{3}}t_\varphi}\sin(q^3/3 - a^2(\varphi)T^{\frac{2}{3}}q)\,dq\,d\varphi.$$

One can extend the path of the q-integral to ∞ introducing an error of $O(T^{-1})$ by the Riemann–Lebesgue Lemma since the only stationary point of the phase is in $[0, T^{\frac{1}{3}}t_\varphi]$

$$I = (T^{-\frac{1}{3}}/2\pi)\int_{\varphi_0}^{\varphi_1} ic_0(\varphi)\int_0^\infty \sin(q^3/3 - a^2(\varphi)T^{\frac{2}{3}}q)dq\,d\varphi.$$

Now the q-integral is a continuous function of φ for φ in the range of integration, and it is a bounded function of T, for all T. To see this one expresses it in terms of the Airy function $Bi(z)$ which is real for z real and is continuous and bounded for all real, non-positive z c.f. [2].

$$\int_0^\infty \sin(q^3/3 - a^2T^{\frac{2}{3}}q)dq = Bi(-a^2T^{\frac{2}{3}}) - \int_0^\infty e^{-q^3/3 - a^2T^{\frac{2}{3}}q}dq.$$

The second term on the right is a continuous function of φ and bounded in T. Hence, since $c_0(\varphi)$ is easily seen to be continuous, $T^{\frac{1}{3}}I$ exists and is bounded in T. Therefore $I = O(T^{-\frac{1}{3}})$.

On the straight section $[u(\varphi)(1 + T^{-\frac{1}{2}}), \infty)$ the u-integral will be $O(T^{-1})$ uniformly in φ $k^{-1} \geqq \cos\varphi \geqq k^{-1} - \delta$, by the Riemann–Lesbesgue Lemma.

Summing up, for $k \geqq 1$, the lowest order contribution to the double integral (A.4) is given by (A.11), i.e. $O(T^{-\frac{1}{3}})$, hence (A.4) will be of that order as $T \to \infty$.

Appendix B

Proof of Theorem 2. We separate (2.3) into two terms; in the first the u-integral is taken on the deformed part, P_d, of the path, and in the second on the straight parts, P_s.

$$\eta(r, \theta) = \int_{-\pi}^{\pi}\int_{P_d} + \int_{-\pi}^{\pi}\int_{P_s}. \tag{B.1}$$

We first show that the second term is of order $O(\log r/r)$. Let φ^* be any value of φ in $[-\pi, \pi]$ such that $\cos(\varphi^* - \theta) = 0$. Then for φ with $|\varphi - \varphi^*| \geqq \delta > 0$, the u-integral will be of order $O(r^{-1})$ uniformly in φ by the Riemann–Lebesgue Lemma; hence so will the double integral. For $|\varphi - \varphi^*| \leqq \delta$ we use the notation

$$I_a^b = \{\varphi \in [-\pi, \pi] | a \leqq |\varphi - \varphi^*| \leqq b\}$$

and the integral becomes, choosing r large enough so that $\delta > r^{-1}$

$$\int_{I_0{}^\delta}\int_{P_s} = \int_{I^\delta{}_{r^{-1}}}\int_{P_s} + \int_{I_0{}^{r^{-1}}}\int_{P_s}. \tag{B.2}$$

The second integral on the right side of (B.2) is of order $O(r^{-1})$ due to the path length of the φ-integral. The first we integrate by parts with respect to u letting $J(u, \varphi)$ denote the coefficient of the exponential function in the integrand. Then writing $\cos(\varphi - \theta) = = \mathrm{sgn}(\theta - \varphi^*)\sin(\varphi - \varphi^*)$ and expanding the analytic functions J and J_u about φ^* one finally gets

$$\int_{I^\delta{}_{r^{-1}}}\int_{P_s} J(u, \varphi)\, e^{iru\cos(\varphi - \theta)}du\,d\varphi$$

$$= \frac{\mathrm{sgn}(\theta - \varphi^*)}{ir}\int_{I^\delta{}_{r^{-1}}}\left\{-\frac{J(0, \varphi^*)}{\sin(\varphi - \varphi^*)} + \frac{J(u(\varphi) - \varepsilon, \varphi^*)\, e^{ir[u(\varphi) - \varepsilon]\cos(\varphi - \theta)}}{\sin(\varphi - \varphi^*)} - \right.$$

$$- \frac{J(u(\varphi) + \varepsilon, \varphi^*)\, e^{ir[u(\varphi)+\varepsilon]\cos(\varphi-\theta)}}{\sin(\varphi - \varphi^*)}$$

$$- \int_{P_s} \frac{J_u(u, \varphi^*)\, e^{iru\cos(\varphi-\theta)}}{\sin(\varphi - \varphi^*)}\, du + R(\varphi - \varphi^*)\Big\}\, d\varphi.$$

Where $R(z)$ is regular at the origin. Now the integral of the first term in brackets is clearly zero. For the second we make the estimate

$$\left| \frac{\operatorname{sgn}(\theta - \varphi^*)}{ir} \int_{I^{\delta}{}_{r^{-1}}} \frac{J(u(\varphi) - \varepsilon, \varphi^*)\, e^{ir[u(\varphi)-\varepsilon]\cos(\varphi-\theta)}}{\sin(\varphi - \varphi^*)}\, d\varphi \right|$$

$$\leqq 2 \operatorname*{Max}_{I^{\delta}{}_{r^{-1}}} |J(u(\varphi) - \varepsilon, \varphi^*)|\, r^{-1} \int_{\varphi^* + r^{-1}}^{\varphi^*+\delta} \frac{d\varphi}{\sin(\varphi - \varphi^*)} = O(\log r/r).$$

In a similar way one shows that the entire integral above is of order $O(\log r/r)$; hence the second term of (B.1) is of that order.

We now consider the first term of (B.1), breaking it up into the four terms:

$$\int_{-\pi}^{\pi} \int_{P_d} = \int_{\substack{-\pi/2 \leqq \varphi \leqq \pi/2 \\ |\varphi - \varphi^*| \geqq \delta}} \int_{P_d} + \int_{\substack{-\pi/2 \leqq \varphi \leqq \pi/2 \\ |\varphi - \varphi^*| \leqq \delta}} \int_{P_d} + \int_{\substack{\pi/2 \leqq |\varphi| \leqq \pi \\ |\varphi - \varphi^*| \geqq \delta}} \int_{P_d} + \int_{\substack{\pi/2 \leqq |\varphi| \leqq \pi \\ |\varphi - \varphi^*| \leqq \delta}} \int_{P_d}$$

$$= I_1 + I_2 + I_3 + I_4.$$

The integrals I_2 and I_4 are of order $O(\delta)$ due to the path length of the φ integral. Consider I_1. The integrand, hence the integral, will go to zero exponentially as $r \to \infty$, if

$$\operatorname{Im}(u) \cos(\varphi - \theta) > 0$$

on the deformed section of the path. For I_1 one has $\operatorname{Im}(u) < 0$ since the path lies below the u axis. Hence for φ such that $\cos(\varphi - \theta) < 0$ the integrand will go to zero. However, for those φ such that $\cos(\varphi - \theta) > 0$, the real part of the exponent is positive and the limiting value of the integrand is not clear. To remedy this, we deform the path of u integration up through the singularity. Then, on the deformed section, one has $\operatorname{Im}(u) > 0$, and the integrand will go to zero as $r \to \infty$. One gets, however, a contribution due to the residue at the pole $u(\varphi)$. Treating I_3 in a similar way and taking δ arbitrarily small one obtains

$$I_1 = \pm i \int_{-\pi/2}^{\pi/2} \frac{H(\varphi) k^2 u^2(\varphi) \cos^2\varphi\, e^{\pm iru(\varphi)\cos(\varphi-\theta)}}{\cosh u(\varphi)[k^2 \cos^2\varphi - \operatorname{sech}^2 u(\varphi)]}\, d\varphi,$$

where $H(\varphi) = \begin{cases} 1 & \text{if } \cos(\varphi - \theta) > 0 \text{ and } u(\varphi) \text{ is defined} \\ 0 & \text{otherwise} \end{cases}$

taking the upper signs. I_3 is given by the lower signs. Adding I_1 and I_3 one obtains an expression for the first term of (B.1):

$$\int_{-\pi}^{\pi} \int_{P_d} = -2 \operatorname{Im} \int_{-\pi/2}^{\pi/2} \frac{H(\varphi) k^2 u^2(\varphi) \cos^2\varphi\, e^{iru(\varphi)\cos(\varphi-\theta)}}{\cosh u(\varphi)[k^2 \cos^2\varphi - \operatorname{sech}^2 u(\varphi)]}\, d\varphi.$$

In view of (2.1) and (2.2) one finally obtains the expression (2.4).

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